Order in Binary Sequences and the Routes to Chaos

Ricardo López-Ruiz

Department of Computer Science and BIFI,
Facultad de Ciencias, Edificio B,
Universidad de Zaragoza,
50009 - Zaragoza (Spain)

February 8, 2008

Abstract

The natural order in the space of binary sequences permits to recover the U-sequence. Also the scaling laws of the period-doubling cascade and the intermittency route to chaos defined in that ordered set are explained. These arise as intrinsic properties of this ordered set, and independent from any consideration about dynamical systems.

Keywords: dynamical systems; symbolic dynamics; orbit generation; information theory; code theory.

1 Introduction

In the last decades many efforts have been devoted to understand and to describe the different routes to chaos that are found in the dynamical behavior of the most diverse physical systems. Many of these situations can be modeled by unimodal maps, which form a family of well understood dynamical systems [1]. Some characteristics of the unimodal maps are gathered in three basic works [2, 3, 4]. Metropolis *et al.* [2] established the order in which the periodic orbits appear in that kind of systems. They found the U*-sequence*, which is a universal sequence of bifurcations common to all unimodal maps. Feigenbaum [3] discovered the constant that brings his name and that is a consequence of the universal scaling properties of the period-doubling cascade. Pomeau and Manneville [4] proposed simple models to understand and to catalog some dynamical mechanisms of bursting in temporal signals.

The fact that so simple systems display so many universal properties is remarkable. In this small piece of work, it is our aim to offer a unified view of a similar panorama happening in the space of the binary sequences. For this purpose, the natural order in the space of binary sequences will be recalled in section 2. Besides the recovering of the U-sequence, the scaling laws for alternative period-doubling cascade and intermittency routes to chaos defined in this binary scheme will be found. These laws are intrinsic properties of the ordered set of binary sequences, and they are also independent from any consideration about dynamical systems.

This optional and simplified view will be presented in section 3. The last section contains the conclusions.

2 Order in the Space of Binary Sequences

Let us recall the nomenclature introduced in Ref. [5]. $\mathcal{B_S}$ represents the space of binary sequences. An orbit of period n (O_n) is formed by a binary sequence non periodic (irreducible) of n digits and all sequences built from the iteration of this irreducible sequence as a block. It is represented by the irreducible orbit. An example can be: $O_3 = 100 \equiv \{100, 100100, 100100100, \ldots\}$. The *real equivalent* of the orbit $O_n = \alpha_{n-1}\alpha_{n-2}\ldots\alpha_1\alpha_0$ is the real number $\sum_{i=0}^{n-1}\alpha_i 2^i$. The set formed by an n-periodic orbit, O_n, and its $n-1$ cyclic permutations will be called *orbital* of period n ($[O_n]$). For example, $[O_3] = [100] \equiv \{100, 010, 001\}$. The set of the conjugate orbits is the *conjugate orbital*: $[\overline{O}_3] = [011] \equiv \{011, 101, 110\}$. The number of orbitals N_n of period n is given by:

$$N_n = \frac{2^n - \sum_{i=1}^{k} m_i N_{m_i}}{n}, \tag{1}$$

where $\{m_1, m_2, \ldots, m_k\}$ are the integer divisors of n excluding $\{n\}$.

Lorenz Order: If $[O_n] \neq [\overline{O}_n]$, the set $L_n^D = [O_n; \overline{O}_n] \equiv [O_n] \cup [\overline{O}_n]$ is called *L-doublet*. If $[O_n] = [\overline{O}_n]$, the set $L_n^S = [O_n;] \equiv [O_n]$ is called *L-singlet*. Given a L-doublet or L-singlet, the orbit that starts in the left side by 1 followed by the subsequence with the smallest real equivalent is called the *characteristic orbit* (O_n^c). This orbit is chosen as representative:

$L_n^D = [O_n^c, \overline{O}_n^c]$, $L_n^S = [O_n^c;]$. For example, if $O_3^c = 100$ then $L_3^D = [100;011]$ or if $O_4^c = 1001$ then $L_4^S = [1001;]$. The *associated fraction* (r) of a L-doublet or a L-singlet is a fraction associated to its characteristic orbit, $O_n^c = \alpha_1\alpha_2\ldots\alpha_n$ and defined as:

$$r = \left(\sum_{i=1}^{n} \frac{\alpha_i}{2^i}\right) \cdot \frac{2^n}{2^n - 1}. \tag{2}$$

These fractions are (except the trivial 1-periodic orbit with $r_{[1;0]} = 1$) in the range $r_{[10;]} = \frac{2}{3} > r > \frac{1}{2} = r_{[100\overset{\infty}{\ldots};011\overset{\infty}{\ldots}]}$.

Now we will establish an order relation in the space of L-doublets and L-singlets, called *implication*, and represented by the symbol $\Rightarrow$. A generic element of this space is represented by the symbol $[;]$. We will said that the element $[;]_i$ implies the element $[;]_j$ when the associated fractions (r_i, r_j) verify $r_i < r_j$:

$$[;]_i \Rightarrow [;]_j \leftrightarrow r_i < r_j. \tag{3}$$

For example, $L_4^D \equiv [1000;0111] \Rightarrow L_3^D \equiv [100;011]$ because $r_{[1000;0111]} = \frac{8}{15} < \frac{4}{7} = r_{[100;011]}$. By applying these rules the ordered binary set showed in table 1 is found. Each element in the table implies all elements above it. We represent the space of binary sequences and the Lorenz-order relationship by $(\mathcal{B_S}, \mathcal{L}_\Rightarrow)$.

<u>Rössler Order</u> (U-sequence): Another order type, which we call Rössler order, is established in $\mathcal{B_S}$. This order generates the U-sequence. It is built from the Lorenz-order with the following steps. We make a different orbit grouping in $(\mathcal{B_S}, \mathcal{L}_\Rightarrow)$: each L-doublet, $L_n^D = [O_n^c; \overline{O}_n^c]$,

and its doubled L-singlet, $L_{2n}^S = [O_n^c \overline{O}_n^c;]$, is grouped in a L-*triplet*, $L_n^T = [O_n^c, \overline{O}_n^c; O_n^c \overline{O}_n^c] = L_n^D \cup L_{2n}^S$. An L-singlet not belonging to any L-triplet is called L-*singlet*$_d$. They have the property: $L_{2n}^{S_d} = [O_n \overline{O}_n;] \rightarrow O_n = O_{\frac{n}{2}} \overline{O}_{\frac{n}{2}}$. Then, $\mathcal{B}_\mathcal{S}$ is divided in a new partition: L-triplets and L-singlets$_d$. We define the binary sequence transformation $\mathcal{F}_{\mathcal{B}_\mathcal{S}}$ that transforms the binary sequence $\{l_1 l_2 \ldots l_n\}$ in the binary sequence $\{r_1 r_2 \ldots r_n\}$ according to the law:

$$\begin{aligned} r_i &= l_i + l_{i+1} \pmod 2 \\ l_{n+1} &= l_1. \end{aligned} \tag{4}$$

The inverse transformation $\mathcal{F}_{\mathcal{B}_\mathcal{S}}{}^{-1}$ is defined as follows:

$$l_i = \sum_{j=0}^{i-1} r_j, \tag{5}$$

with $r_0 = 1$ and $r_{n+i} = r_i$. This last transformation must be applied to the doubled sequence $\{r_1 r_2 \ldots r_n r_1 r_2 \ldots r_n\}$ in order to get the doubled sequence $\{l_1 l_2 \ldots l_n l_1 l_2 \ldots l_n\}$.

If we apply $\mathcal{F}_{\mathcal{B}_\mathcal{S}}$ to the characteristic orbit O_n^c of a L-triplet the result is the regular-orbit O_n^r, an orbit with even number of 1s. If $\mathcal{F}_{\mathcal{B}_\mathcal{S}}$ is applied to the orbit $O_n^c \overline{O}_n^c$ of the same L-triplet the result is the flip-orbit O_n^f, an orbit with odd number of 1s. Thus the L-triplet is transformed by $\mathcal{F}_{\mathcal{B}_\mathcal{S}}$ into two independent orbitals whose union, $R_n^D = [O_n^r; O_n^f] \equiv [O_n^r] \cup [O_n^f]$, is called an R-*doublet*. The transformation of the L-singlet$_d$ $L_{2n}^{S_d}$ by $\mathcal{F}_{\mathcal{B}_\mathcal{S}}$ produce an orbital called an R-*singlet*, R_n^S. The action of $F_{S_\mathcal{B}}$ is summarized as follows:

$$\mathcal{F}_{\mathcal{B}_\mathcal{S}} : \quad \mathcal{B}_\mathcal{S} \quad \longrightarrow \quad \mathcal{B}_\mathcal{S}$$

$$L_n^T \left\{ \begin{array}{ccc} L_n^D & \longrightarrow & [O_n^r] \\ L_n^S & \longrightarrow & [O_n^f] \end{array} \right\} R_n^D$$
$$L_{2n}^{S_d} \longrightarrow R_n^S$$

Moreover, $\mathcal{F}_{\mathcal{B}_\mathcal{S}}$ transfers the Lorenz-order in $(\mathcal{B}_\mathcal{S},\ \mathcal{L}_\Rightarrow)$ to a new order called R-order (Table 2). This order relationship is defined as follows: if $L_{n_i}^{S_d,T} \Rightarrow L_{n_j}^{S_d,T}$ then the transformed orbitals by $\mathcal{F}_{\mathcal{B}_\mathcal{S}}$ verify that $R_{n_i}^{D,S} \Rightarrow R_{n_j}^{D,S}$. (This is well-defined because the L-doublet and the L-singlet comprising the L-triplet do not have any other orbital between them in the L-order). The new ordered space is denoted by $(\mathcal{B}_\mathcal{S},\ \mathcal{R}_\Rightarrow)$. This is the U-sequence (see Table 1).

3 Routes to Chaos

An infinite subset $\mathcal{C}$ of the ordered set $(\mathcal{B}_\mathcal{S}, \mathcal{L}_\Rightarrow)$ is called a *route to chaos* if $([;]_i, [;]_j) \in \mathcal{C}$ and $[;]_i \Rightarrow [;]_k \Rightarrow [;]_j$ implies that $[;]_k \in \mathcal{C}$. There are in the ordered set $(\mathcal{B}_\mathcal{S}, \mathcal{L}_\Rightarrow)$ two important kinds of route to chaos with their respective scaling laws that we proceed to present now.

Period-doubling route to chaos, $\mathcal{C}_{\mathcal{PD}}$: This set is formed by a L-doublet, $L_n^D = [O_n^c; \overline{O}_n^c]$, and all the consecutive L-singlets of double period. That is $\mathcal{C}_{\mathcal{PD}} \equiv \{L_n^D, L_{2n}^S, L_{4n}^S, \ldots, L_\infty^S\}$. For example, $\{1, 10, 1001, 10010110, \ldots\}$.

Let us calculate the sequence of associated fractions $\{r_n, r_{2n}, r_{4n}, \ldots, r_\infty\}$.

A straightforward calculation gives us:

$$r_{2n} = [r_n(2^n - 1) + 1] \frac{2^n - 1}{2^{2n} - 1}. \tag{6}$$

The scaling law associated to this route to chaos is:

$$\lim_{n \to \infty} \frac{2^{2n}(r_n - r_{2n})}{2^{4n}(r_{2n} - r_{4n})} = 1. \tag{7}$$

This law is independent of the particular set $\mathcal{C}_{\mathcal{PD}}$ and it is intrinsic to the own structure of the ordered space of binary sequences. (This is reflected in the Feigenbaum constant for unimodal maps with quadratic critical point).

<u>*Intermittency route to chaos*</u>, $\mathcal{C}_{\mathcal{IT}}$: Let us take any L-doublet expressed by $L^D_p = [O_p; \overline{O}_p]$ where $[O_p] = [\alpha_1\alpha_2 \ldots \alpha_p]$. We define the orbital $[O_{p+1}] = [\alpha_1\alpha_2 \ldots \alpha_p 1]$ and the L-doublet associated to it, i.e., $L^D_{p+1} = [O_{p+1}, \overline{O}_{p+1}]$. The route to chaos $\mathcal{C}_{\mathcal{IT}}$ defined by these two extrema L-doublets: $\mathcal{C}_{\mathcal{IT}} \equiv \{L^D_{p+1}, \ldots, L^D_p\}$ is called the intermittency route to chaos. An example can be: $\{[1001;], \ldots, [100; 011]\}$. We choose a subsequence of $\mathcal{C}_{\mathcal{IT}}$ in the following form:

$$\begin{array}{rcl}
L^D_{p+1} = [O_{p+1}, \overline{O}_{p+1}] & \rightarrow & O_{p+1} = \alpha_1 \ldots \alpha_p 1 \\
L^D_{2p+1} = [O_{2p+1}, \overline{O}_{2p+1}] & \rightarrow & O_{2p+1} = \alpha_1 \ldots \alpha_p \alpha_1 \ldots \alpha_p 1 \\
\vdots & \vdots & \vdots \\
L^D_{np+1} = [O_{np+1}, \overline{O}_{np+1}] & \rightarrow & O_{np+1} = \alpha_1 \ldots \alpha_p \overset{(n)}{\cdots} \alpha_1 \ldots \alpha_p 1 \\
\vdots & \vdots & \vdots \\
L^D_p = [O_p, \overline{O}_p] & \rightarrow & O_p = \alpha_1 \ldots \alpha_p
\end{array}$$

The subsequence from the last example is: $\{1001, 1001001, 1001001001, \ldots, 100100 \overset{100}{\cdots} 1001, \ldots, 100\}$. Given the fraction r_p associated to L_p^D, a straightforward calculation permits to find the associated fractions of the remaining fraction subsequence:

$$r_{np+1} = \frac{2r_p(2^{np} - 1) + 1}{2^{np+1} - 1}. \tag{8}$$

Then, the subsequence $\{r_{p+1}, r_{2p+1}, \ldots, r_{np+1}, \ldots, r_p\}$ is obtained. We denote by $\delta = r_{np+1} - r_p$ the distance to the critical fraction r_p where the periodic behaviour take place and the integer $l = np$ of an orbit O_{np+1} represents the binary length of the periodic sequence of this orbit ('phase laminar' length). The isolated digit 1 that breaks this periodic behavior is called a 'burst'. If we calculate the relation between δ and l, the scaling law for the intermittency route to chaos is obtained:

$$r_{np+1} - r_p = \delta \sim \frac{1 - r_p}{2^{np+1}} \Longrightarrow 4\delta \cdot 2^l \sim 1, \tag{9}$$

where r_p has been taken as 0.5. Let us observe that when the distance to the critical point goes to zero, we recover the divergence of laminar phase length as an intrinsic and universal property of the ordered set of binary sequences.

4 Conclusions

The properties of the routes to chaos in high dimensional systems is still a not very well understood subject [6]. Unidimensional results are generalized for instance in the work [7], where an orbit implication diagram for

horseshoe type-flows is calculated by topological methods. This provides a partial order on orbit formation for three dimensional flows and two dimensional orientation preserving maps which evolve into horseshoe under parameter variation. The results are independent of dissipation from the conservative limit (zero dissipation) to the unimodal limit (infinite dissipation).

In this work, two routes to chaos have been defined in the space of binary sequences. One of them mimics the period-doubling cascade arising in the unimodal maps and the other one mimics the intermittency route to chaos. The scaling properties for these both routes to chaos have been calculated. A Feigenbaum-like relationship for the bifurcation parameters in the period doubling case and the dependence of the laminar phases length on the distance to the critical point in the intermittency scenario seem to be a consequence of the intrinsic properties of the ordered binary set, and independent from any consideration about dynamical systems. This simple and primitive view of the route to chaos could be interpreted as the *radiograph* of these dynamical phenomena when they are observed in more complicated systems.

Acknowledgements: This problem comes from Philadelphia (1993). I must thank Prof. R. Gilmore (Philadelphia) and Prof. G. Mindlin (Buenos Aires) for very interesting discussions.

References

[1] P. Collet and J.P. Eckmann, *Iterated maps on the interval as dynamical systems*, Birkhaüser, Boston (1980).

[2] N. Metropolis, M.L. Stein and P.R. Stein, *On finite limit sets for transformations on the unit interval*, J. Combinatorial Theory **15**:25-44 (1973).

[3] M.J. Feigenbaum, *Quantitative universality for a class of nonlinear transformations*, J. Statist. Phys. **19**:25-52 (1978).

[4] Y. Pomeau and P. Manneville, *Intermittent transition to turbulence in dissipative dynamical systems*, Commun. Math. Phys. **74**:189-197 (1980).

[5] R. López-Ruiz, *A binary approach to the Lorenz model*, Chaos, Solitons and Fractals **8**, 1-6 (1997).

[6] D. Fournier-Prunaret, R. López-Ruiz and A. Taha, *Route to chaos in three-dimensional maps of logistic map*, ECIT'04-ITERATION THEORY, preprint nlin.CD/0502012 (2005).

[7] G.B. Mindlin, R. López-Ruiz, H.G. Solari and R. Gilmore, *Horseshoe implications*, Phys. Rev. E **48**:4297-4304 (1993).

Table Captions

Table 1. Rössler-order ($\mathcal{R}_{\Rightarrow}$) and Lorenz-order ($\mathcal{L}_{\Rightarrow}$) in the set of binary sequences until period $n = 6$. Also the order of the orbits in unimodal maps is given. (* means period-doubled orbits).

Table 2. The two different orbit groupings, Rössler and Lorenz type, established in the space of binary sequences.

$R-Order$	$Period_{order}$	$L-Order$	r	r
0	1_1	1 ; 0	1	1.00000
1		10	2/3	0.66666
10	2_1^*	1001	3/5	0.60000
1011	4_1^*	10010110	10/17	0.58823
101110	6_1	100101 ; 011010	37/63	0.58730
101111		10010101101	2394/4095	0.58461
10111	5_1	10010 ; 01101	18/31	0.58064
10110		1001001101	589/1023	0.57575
101	3_1	100 ; 011	4/7	0.57142
100		100011	35/63	0.55555
100101	6_2^*	100011011100	2268/4095	0.55384
10010	5_2	10001 ; 01110	17/31	0.54838
10011		1000101110	558/1023	0.54545
100111	6_3	100010 ; 011101	34/63	0.53968
100110		100010011101	2205/4095	0.53846
1001	4_2	1000 ; 0111	8/15	0.53333
1000		10000111	135/63	0.52941
100010	6_4	100001 ; 011110	11/21	0.52380
100011		100001011110	2142/4095	0.52307
10001	5_3	10000 ; 01111	16/31	0.51612
10000		1000001111	527/1023	0.51515
100001	6_5	100000 ; 011111	32/63	0.50793
100000		100000011111	2079/4095	0.50769

Table 1:

Lorenz Grouping	*Rössler Grouping*
Binary Sequences	Binary Sequences
Orbits, O_n Orbitals, $[O_n]$	Orbits, O_n Orbitals, $[O_n]$
L-doublet, L_n^D L-singlet, L_n^S	
L-triplet, L_n^T L-singlet$_d$, $L_{2n}^{S_d}$	R-doublet, R_n^D R-singlet, R_n^S

Table 2: